

From the Pseudosphere to the Mirror Amsler Surface: Interactive Visualization of Bäcklund Deformation Scenarios for $K = -1$ Surfaces

This record contains a self-contained interactive HTML visualization of the Bäcklund deformation connecting the pseudosphere with the Amsler surface along a path in the parameter plane (s, a) . The record is a numerical and visual demonstration, not a proof. For reference, the theorem that the demonstration illustrates is stated below; its complete rigorous proof, including all uniformity estimates, will be given in a forthcoming publication.

Theorem demonstrated (proof to appear separately). Let $f_1(x, y) = Y(xy)$ be the Amsler seed, where Y solves

$$(uY')' = \sin Y, \quad Y(0) = \pi s, \quad s \in (0, 1).$$

Let $p = p(s, a, t_0)$ be the solution of the Bäcklund system

$$p_x = \frac{1}{a} \sin(f_1 - p), \quad p_y = f_{1y} - a \sin p,$$

with Goursat data

$$p(x, 0) = y_0 - 2 \arctan(t_0 e^{-x/a}), \quad p(0, y) = 2 \arctan(\tan(p_{00}/2) e^{-ay}),$$

where $y_0 = \pi s$ and $p_{00} = y_0 - 2 \arctan t_0$. Set $\omega(s, a, t_0) = 2p - f_1$, and let $r(s, a, t_0)$ be the Lelièvre immersion of ω with a fixed initial frame. Then:

- (*Continuity*) The map $(s, a, t_0) \mapsto \omega(s, a, t_0) \in C(W)$ is continuous on $(0, 1) \times [1, \infty) \times (0, \infty)$, and the induced immersions depend continuously on (s, a, t_0) .
- (*Pseudospherical kink limits*) As $s \rightarrow 1$ and as $s \rightarrow 0$, $\omega(s, a, t_0)$ converges uniformly on W to the explicit pseudospherical kink solutions

$$\omega = 2 \operatorname{gd}(z_0 + x/a - ay) \quad \text{and} \quad \omega = 4 \arctan(\tan(\omega(0, 0)/4) e^{-(x/a + ay)}),$$

respectively, where gd is the Gudermannian function.

- (*Amsler limit*) For fixed $s \in (0, 1)$ and $t_0 > 0$, as $a \rightarrow \infty$ the immersions $r(s, a, t_0)$ converge uniformly on W to the Lelièvre immersion of the mirror Amsler surface $\omega_{\text{ref}} = -f_1$, with rate $O(1/a)$; the constant is uniform in $t_0 > 0$ and on compact subsets of $s \in (0, 1)$.

The theorem is stated here only to identify what the demonstration visualizes. No part of the proof is contained in this record, and the record should not be cited as establishing the theorem. The forthcoming paper is the reference for rigorous statements and proofs.

What the demonstration shows. The visualization is centred on the deformation path from the pseudospherical endpoint $(s, a) = (0, 1)$ to the near-Amsler endpoint $(s, a) = (1/2, 10^4)$. Two scenarios are provided: a two-phase path, where (s, a) moves simultaneously from $(0, 1)$ to $(1/2, 10^4)$; and a sequential path, where first $s : 0 \rightarrow 1/2$ at fixed $a = 1$, then $a : 1 \rightarrow 10^4$ at fixed $s = 1/2$. Additional controls allow sweeping a at fixed s , looping s near the exact kink limits, changing the edge-shift parameter t_0 , the window half-width L , and the zoom. The surface is reconstructed from $\omega = 2p - f_1$ by marching the Gauss–Weingarten frame. The cuspidal edge $\sin \omega = 0$ is highlighted; the colour encodes the value of ω . Numerical diagnostics display the sine-Gordon residual, frame drift, the number of cuspidal lines in the window, and the distance to the exact pseudospherical kink. The scheme is stable over the displayed range $a \in [0.4, 10^4]$. The record is intended for researchers and students interested in integrable surfaces, Bäcklund transformations, constant negative curvature immersions, and the Amsler–pseudosphere correspondence. It runs directly in a browser without installation.